Interval Prediction for Continuous-Time Systems with Parametric Uncertainties

Edouard Leurent[1,2], Denis Efimov[1],
Tarek Raïssi[3], Wilfrid Perruquetti[4]

[1] Inria, Lille, France
[2] Renault Group, Guyancourt, France
[3] CNAM, Paris, France
[4] Centrale Lille, France

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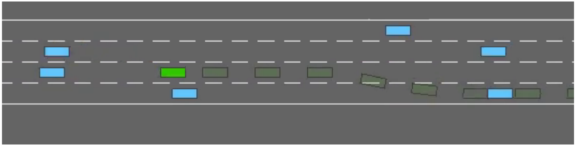
Problem statement

We are interested in trajectory planning for an autonomous vehicle.



1. We need to predict the behaviours of other drivers
2. These behaviours are uncertain and non-linear

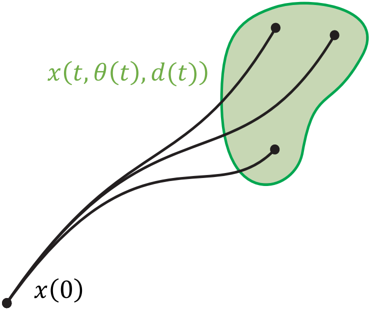
In order to efficiently capture model uncertainty, we consider the modelling framework of Linear Parameter-Varying systems.

Linear Parameter-Varying systems

$$\dot{x}(t) = A(\theta(t))x(t) + Bd(t)$$

There are two sources of uncertainty:

- Parametric uncertainty $\theta(t)$
- External perturbations $d(t)$

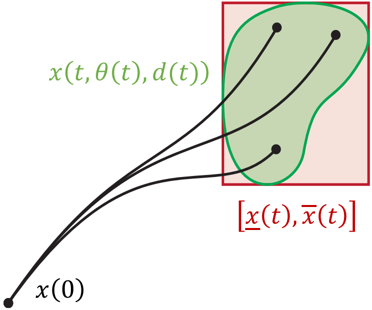


Interval Prediction

Can we design an interval predictor $[\underline{x}(t), \overline{x}(t)]$ that verifies:

- inclusion property: $\forall t, \underline{x}(t) \leq x(t) \leq \overline{x}(t)$;
- stable dynamics?

We want the predictor to be as tight as possible.



Assumption (Bounded trajectories)

- $\|x\|_\infty < \infty$
- $x(0) \in [\underline{x}_0, \overline{x}_0]$ *for some known* $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$

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Assumption (Bounded parameters)

- $\theta(t) \in \Theta$ *for some known* Θ
- *The matrix function* $A(\theta)$ *is known*

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Assumption (Bounded perturbations)

- $d(t) \in [\underline{d}(t), \overline{d}(t)]$ *for some known signals* $\underline{d}, \overline{d} \in \mathcal{L}_\infty^n$

How to proceed?

A first idea

Assume that $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$, for some $t \geq 0$.

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↳ Why not use interval arithmetics?

Lemma (Image of an interval (Efimov et al. 2012))

If A a known matrix, then

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}.$$

where $A^+ = \max(A, 0)$ and $A^- = A - A^+$.

Assume that $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$, for some $t \geq 0$.

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↳ Why not use interval arithmetics?

Lemma (Product of intervals (Efimov et al. 2012))

If A is unknown but bounded $\underline{A} \leq A \leq \overline{A}$,

$$\underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- \leq Ax$$
$$\leq \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^-.$$

Assume that $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$, for some $t \geq 0$.

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✓ Since $A(\theta)$ and the set Θ are known,
we can easily compute such bounds $\underline{A} \leq A(\theta(t)) \leq \overline{A}$

Following this result, define the predictor:

$$\begin{aligned}
\dot{\underline{x}}(t) &= \underline{A}^{+}\underline{x}^{+}(t) - \overline{A}^{+}\underline{x}^{-}(t) - \underline{A}^{-}\overline{x}^{+}(t) \\
&\quad + \overline{A}^{-}\overline{x}^{-}(t) + B^{+}\underline{d}(t) - B^{-}\overline{d}(t), \\
\dot{\overline{x}}(t) &= \overline{A}^{+}\overline{x}^{+}(t) - \underline{A}^{+}\overline{x}^{-}(t) - \overline{A}^{-}\underline{x}^{+}(t) \\
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\end{aligned} \tag{1}$$

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Proposition (Inclusion property)

✓ *The predictor* (1) *satisfies* $\underline{x}(t) \leq x(t) \leq \overline{x}(t)(t)$

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\dot{\overline{x}}(t) &= \overline{A}^+\overline{x}^+(t) - \underline{A}^+\overline{x}^-(t) - \overline{A}^-\underline{x}^+(t) \\
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? But is it stable?

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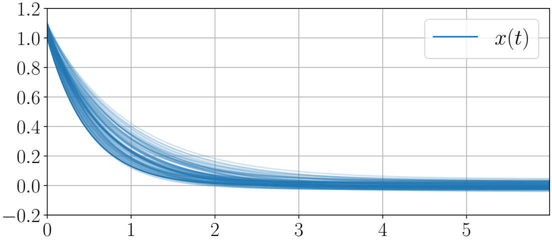
Consider the scalar system, for all $t \geq 0$:

$$\dot{x}(t) = -\theta(t)x(t) + d(t), \text{ where } \begin{cases} x(0) \in [\underline{x}_0, \overline{x}_0] = [1.0, 1.1], \\ \theta(t) \in \Theta = [\underline{\theta}, \overline{\theta}] = [1, 2], \\ d(t) \in [\underline{d}, \overline{d}] = [-0.1, 0.1], \end{cases}$$



✓ The system is always stable

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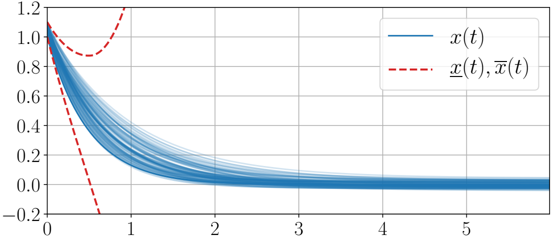

✓ The system is always stable ✗ The predictor (1) is unstable

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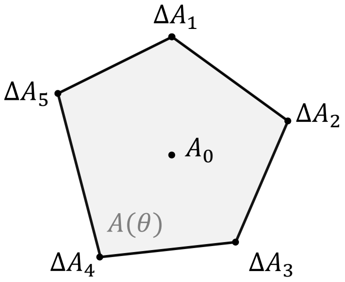
Our proposed predictor

Assumption (Polytopic Structure)

There exist A_0 *Metzler and* $\Delta A_0, \cdots, \Delta A_N$ *such that:*

$$A(\theta) = \underbrace{A_0}_{\substack{\text{Nominal}\\\text{dynamics}}} + \sum_{i=1}^{N} \lambda_i(\theta)\Delta A_i, \quad \sum_{i=1}^{N} \underbrace{\lambda_i(\theta)}_{\geq 0} = 1; \quad \forall \theta \in \Theta$$



Denote

$$\Delta A_+ = \sum_{i=1}^{N} \Delta A_i^+, \ \Delta A_- = \sum_{i=1}^{N} \Delta A_i^-,$$

We define the predictor

$$\begin{aligned}
\dot{\underline{x}}(t) &= A_0\underline{x}(t) - \Delta A_+\underline{x}^-(t) - \Delta A_-\overline{x}^+(t) \\
&\quad +B^+\underline{d}(t) - B^-\overline{d}(t), \\
\dot{\overline{x}}(t) &= A_0\overline{x}(t) + \Delta A_+\overline{x}^+(t) + \Delta A_-\underline{x}^-(t) \\
&\quad +B^+\overline{d}(t) - B^-\underline{d}(t), \\
&\quad \underline{x}(0) = \underline{x}_0, \ \overline{x}(0) = \overline{x}_0
\end{aligned} \tag{2}$$

Theorem (Inclusion property)

The predictor (2) *ensures* $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$.

Theorem (Stability)

If there exist diagonal matrices P, Q, Q_+, Q_-, Z_+, Z_-, Ψ_+, Ψ_-, Ψ, $\Gamma \in \mathbb{R}^{2n\times 2n}$ such that the following LMIs are satisfied:

$$P + \min\{Z_+, Z_-\} > 0, \ \Upsilon \preceq 0, \ \Gamma > 0,$$
$$Q + \min\{Q_+, Q_-\} + 2\min\{\Psi_+, \Psi_-\} > 0,$$

where $\Upsilon = \Upsilon(A_0, \Delta A_-, \Delta A_+, \Psi_-, \Psi_+, \Psi)$,
then the predictor (2) *is input-to-state stable with respect to the inputs* $\underline{d}$, $\overline{d}$.

Sketch of proof

1. Define the extended state vector as $X = [\underline{x}^\top \ \ \overline{x}^\top]^\top$
2. It follows the dynamics

$$\dot{X}(t) = \mathcal{A}X(t) + R_+X^+(t) - R_-X^-(t) + \delta(t)$$

$$\mathcal{A} = \begin{bmatrix} A_0 & 0 \\ 0 & A_0 \end{bmatrix} \; R_+ = \begin{bmatrix} 0 & -\Delta A_- \\ 0 & \Delta A_+ \end{bmatrix}, \; R_- = \begin{bmatrix} \Delta A_+ & 0 \\ -\Delta A_- & 0 \end{bmatrix}$$

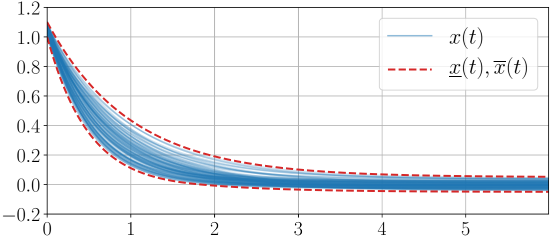
3. Consider a candidate Lyapunov function:

$$V(X) = X^\top P X + X^\top Z_+ X^+ - X^\top Z_- X^-$$

4. $V(X)$ is positive definite provided that $P + \min\{Z_+, Z_-\} > 0$,
5. Check on which condition we have $\dot{V}(X) \leq 0$

Recall the scalar system:

$$\dot{x}(t) = -\theta(t)x(t) + d(t), \text{ where } \begin{cases} x(0) \in [\underline{x}_0, \overline{x}_0] = [1.0, 1.1], \\ \theta(t) \in \Theta = [\underline{\theta}, \overline{\theta}] = [1, 2], \\ d(t) \in [\underline{d}, \overline{d}] = [-0.1, 0.1], \end{cases}$$

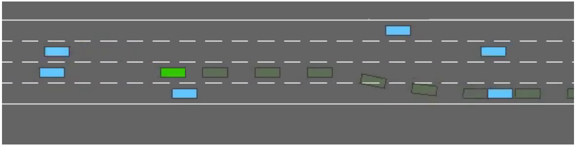


✓ The system is always stable

✓ The predictor (2) is stable

03

Application to autonomous driving



$$\dot{z}_i = f_i(Z, \theta_i),\ i = \overline{1, N},$$

where

- $z_i = [x_i, y_i, v_i, \psi_i]^\top \in \mathbb{R}^4$ is the state of an agent
- $\theta_i \in \mathbb{R}^5$ is the corresponding unknown behavioural parameters
- $Z = [z_1, \dots, z_N]^\top \in \mathbb{R}^{4N}$ is the joint state of the traffic
- $\theta = [\theta_1, \dots, \theta_N]^\top \in \Pi \subset \mathbb{R}^{5N}$

States	(x_i, y_i)	position
	v_i	longitudinal velocity
	ψ_i	yaw angle
Controls	a_i	longitudinal acceleration
	β_i	slip angle at the center of gravity

Each vehicle follows the Kinematic Bicycle Model:

$$\begin{aligned}
\dot{x}_i &= v_i \cos(\psi_i),\\
\dot{y}_i &= v_i \sin(\psi_i),\\
\dot{v}_i &= a_i,\\
\dot{\psi}_i &= \frac{v_i}{l} tan(\beta_i),
\end{aligned}$$

A linear controller using three features inspired from the intelligent driver model (IDM) [Treiber et al. 2000].

$$a_i = \begin{bmatrix}\theta_{i,1} & \theta_{i,2} & \theta_{i,3}\end{bmatrix} \begin{bmatrix} v_0 - v_i \\ -(v_{f_i} - v_i)^- \\ -(x_{f_i} - x_i - (d_0 + v_i T))^- \end{bmatrix},$$

where

v_0	speed limit
d_0	jam distance
T	time gap
f_i	index of vehicle *i*'s front vehicle

A cascade controller of lateral position y_i and heading ψ_i:

$$\dot{\psi}_i = \theta_{i,5}\left(\psi_{L_i} + \sin^{-1}\left(\frac{\widetilde{v}_{i,y}}{v_i}\right) - \psi_i\right), \quad (3)$$

$$\widetilde{v}_{i,y} = \theta_{i,4}(y_{L_i} - y_i).$$

We assume that the drivers choose their steering command β_i such that (3) is always achieved: $\beta_i = \tan^{-1}(\frac{l}{v_i}\dot{\psi}_i)$.

We linearize trigonometric operators around $y_i = y_{L_i}$ and $\psi_i = \psi_{L_i}$. This yields the following longitudinal dynamics:

$$\dot{x}_i = v_i,$$
$$\dot{v}_i = \theta_{i,1}(v_0 - v_i) + \theta_{i,2}(v_{f_i} - v_i) + \theta_{i,3}(x_{f_i} - x_i - d_0 - v_i T),$$

where $\theta_{i,2}$ and $\theta_{i,3}$ are set to 0 whenever the corresponding features are not active.

$$\dot{Z} = A(\theta)(Z - Z_c) + d.$$

For example, in the case of two vehicles only:

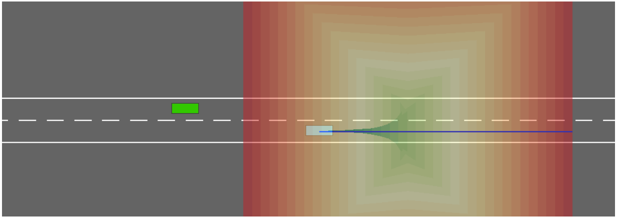
$$Z = \begin{bmatrix} x_i \\ x_{f_i} \\ v_i \\ v_{f_i} \end{bmatrix}, \quad Z_c = \begin{bmatrix} -d_0 - v_0 T \\ 0 \\ v_0 \\ v_0 \end{bmatrix}, \quad d = \begin{bmatrix} v_0 \\ v_0 \\ 0 \\ 0 \end{bmatrix}$$

$$A(\theta) = \begin{array}{c} \\ i \\ f_i \\ i \\ f_i \end{array} \begin{array}{c} \begin{array}{cccc} i & f_i & i & f_i \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\theta_{i,3} & \theta_{i,3} & -\theta_{i,1} - \theta_{i,2} - \theta_{i,3} & \theta_{i,2} \\ 0 & 0 & 0 & -\theta_{f_i,1} \end{bmatrix} \end{array}$$

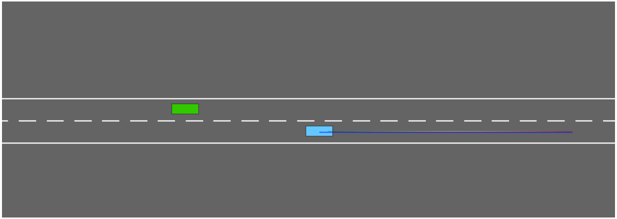
The lateral dynamics are in a similar form:

$$\begin{bmatrix} \dot{y}_i \\ \dot{\psi}_i \end{bmatrix} = \begin{bmatrix} 0 & v_i \\ -\frac{\theta_{i,4}\theta_{i,5}}{v_i} & -\theta_{i,5} \end{bmatrix} \begin{bmatrix} y_i - y_{L_i} \\ \psi_i - \psi_{L_i} \end{bmatrix} + \begin{bmatrix} v_i \psi_{L_i} \\ 0 \end{bmatrix}$$

Here, the dependency in v_i is seen as an uncertain parametric dependency, *i.e.* $\theta_{i,6} = v_i$, with constant bounds assumed for v_i using an overset of the longitudinal interval predictor.

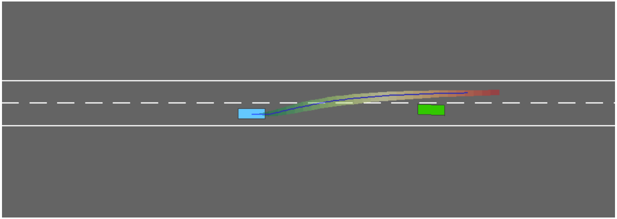


The naive predictor (1) quickly diverges

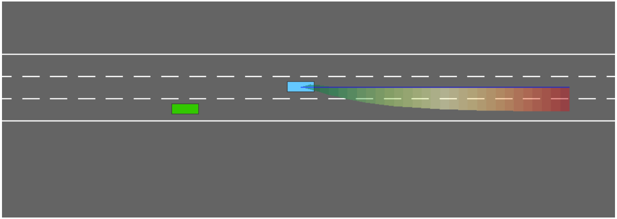


The proposed predictor (2) remains stable

Results



Prediction during a lane change maneuver



Prediction with uncertainty in the followed lane L_i

Problem formulation

- Prediction of an uncertain non-linear system
- ↳ Within the LPV framework
- ↳ Design of an interval predictor $[\underline{x}(t), \overline{x}(t)]$?

Proposed solution

- Direct prediction with interval arithmetics is valid but unstable
- ↳ Assume polytopic uncertainty structure around a nominal A_0
- ↳ Ensure stability using a Luyapunov function in an LMI form

Application

- Joint prediction of coupled traffic dynamics
- Can be used as a building block for robust planning